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Evolution of the Significant Figure Rules

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Today, almost all introductory physics textbooks include standardized "rules" on how to find the number of significant figures in a calculated value.[1-6] And yet, 30 years ago these rules were almost nonexistent. Why have we increased the role of significant figures in introductory classes, and should we continue this trend? A look back at the evolution of significant figures over the last 300 years, from Newton to Millikan to modern authors, sheds some light on their purpose moving forward. While there is much discussion for[7,8] and against[9,10] their use, especially in chemistry, a review of earlier versions of the rules suggests that we have lost some items of value, most notably, a significant figure rule for angles. In addition, we have lost the emphasis that the significant figure rules were designed to calculate an approximate (not exact) precision. Now that the significant figure rules are ingrained into our introductory physics sequence, we would be wise to reiterate that these are just general "rules of thumb."

Evolution and origins of significant figures

While one might think that the origins of the significant figure rules would coincide with the introduction of the calculator, the story of significant figures actually begins in the 1700s, when the phrase had a completely different meaning from the modern usage. Originally, a significant figure was any digit 1 through 9.[11,12] Leading or trailing zeros were not considered significant and were just place holders to locate the decimal. Notably, this early definition is used by Sir Isaac Newton to explain some interesting properties of multiplication.[13] Newton notes that when two numbers are multiplied, "the Number of significant Figures in the Product...can never be greater than the Number of significant Figures in both Factors.... Thus $0.0024 \times 0.03 = 0.000072$." According to this definition, both the 7 and the 2 in the calculation of 0.000072 would be significant. The modern definition, as we shall see, would indicate that only the 7 would be significant. Interestingly, this early definition is concerned exclusively with numerical digits and does not deal with measurement at all.

A. Update to Newtonian notion of significant figures

It is not until the 1800s that the modern definition of significant figures replaces the one described by Newton. In the modern definition, significant figures are digits that have been reliably measured and indicate the precision of the value. This usage appears in an 1888 essay[14] by Silas Whitcomb Holman and also later in his 1892 textbook *Discussion of the Precision of Measurements*.[15] Holman explains: "By a significant figure is here meant any of the 10 digits except such zeros as are inserted merely to enable the point to be located.... In 206.70400 there would be nothing to show whether the last two zeros were significant or useless, but adopting the above definition... they would be significant figures.... This definition of the term 'significant figure' is not in accordance with that sometimes given which limits the meaning to the nine digits other than zero."

Holman also lays out a series of "Rules for Significant Figures" that allow students to perform calculations while retaining the precision of the originally measured values. In the introduction to the 1892 text,[15] he hails the work as "the outcome of an effort to establish such a procedure which, while being sufficiently general, shall not be too laborious in its operation." He notes that the rules are intended to "be utilized without any employment whatever of the calculus, so that they may be applied by one who has forgotten his earlier knowledge of that subject or who has never become acquainted with it."[15] The early date of the text and the careful description of significant figures suggests that Holman may be the first to enumerate the significant figure rules.

However, the idea that the precision of a calculation is set by the significant figures in the input values is certainly older than Holman's description. Manseld Merriman in his 1879 article "Notes on Logarithms"[16] remarks that: "As a general rule it is often said that the number of accurate significant figures in the result of a computation is the same as the number of decimals in the logarithms employed." Henry Raper, in his preface to the 1840 edition on *The Practice of Navigation and Nautical Astronomy*[17] berated would-be seamen at length on calculating numbers to a "useless" precision. In a series of itemized remarks he states: "In using logarithms it is proper to observe that the number (whether it contain decimals or not), and the decimal part of the logarithm, are in general true to the same number of figures.... This remark should be kept in view, because it is mere waste of time to employ more figures than are required in the practical result." Thus, there is an older tradition of generalities that may be classified as significant figure rules; Holman's contribution is to clearly define these rules for student use.

While Holman's significant figure rules were composed at the end of the 19th century, introductory physics texts in the early 1900s were silent on the subject.[18-21] A notable exception is a physics text by the popular author and Nobelist Robert Millikan. Millikan's text *Mechanics, Molecular Physics, Heat, and Sound,* published in 1937,[22] has an appendix on "Significant Figures and Notations by Powers of Ten" that is surprisingly similar in both length and form to the way significant figure rules are presented in modern texts, though the rules themselves are different from the modern incarnation. Interestingly, Millikan's significant figure rules are also

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distinct from Holman's and may be derived from several sources.[14,23-25] However, aside from Millikan's 1937 text, the majority of physics texts composed before 1980 do not include significant figure rules.[26-28]

B. Influence of the slide rule and calculator

The apparent lack of significant figure rules in physics texts in the first half of the 20th century is most likely due to the popularity of the slide rule. The slide rule is a device that allows the user to perform a calculation (multiplication, division, square root, etc.) for two values by sliding a scaled ruler (Fig. 1). The number of significant figures in the computed answer is limited by the precision of the ruler, typically to three digits. In the preface to the introductory physics text *Physics in Two Volumes* published in 1974,[26] Donald Ivey explains as much: "The numbers are invented and I can invent them to the degree of accuracy I please. In general I shall use simple numbers, so that calculations will not be messy, and the numbers are to be assumed as exact, unless otherwise indicated. In general, calculations should be carried out with slide rule accuracy, and answers will be given to three figures." As the slide rule limits the precision of calculations, this eliminates the need for the significant figure rules in solving textbook problems. Moreover, this practice of giving answers to three figures is still carried out in modern texts[1,4] and may be a remnant of the slide rule era.

Fig. 1. Before calculators, computations were done with a slide rule. A close-up of the scale on a slide rule shows the precision of the device to be three significant figures.

The explosion of significant figure rules in physics texts in the 1980s[29-31] is almost certainly due to the widespread use of the calculator. Texts that mention the slide rule do not have a section devoted to significant figures,[26,28] while texts that mention the calculator do have a significant figures section.[29-31] In Sears and Zemansky's popular text *University Physics*, the authors remark in the 1982 edition,[30] "You may do the arithmetic with a calculator having a display with five to 10 digits. But you should not give a 10-digit answer for a calculation using numbers with three significant figures. To do so is not only unnecessary but it is also genuinely wrong because it misrepresents the precision of the results." However, in the 1970 edition[28] the authors have the opposite problem. In 1970 they need to remind students to give more precision, and in one particular problem they remark, "[The] computation must be carried out to four significant figures, which is beyond the precision of an ordinary slide rule and requires the use of four-place logarithms." Thus, the introduction of the significant figure rules into physics texts is most likely a response to careless students writing down the 10 digits shown on their calculator.

C. Standardization of the modern significant figure rules

Today many introductory textbooks include a standardized set of significant figure rules for addition and multiplication in one of the first chapters of the book.[1-6] These significant figure rules, unlike previous formats,[14,22-25,29-31] are completely uniform. This is most likely due to the standardization of the rules in 1993 by the American Society for Testing and Materials (ASTM).[32] This standardization has not only increased the relevance of the significant figure rules, but may have also cemented their place in introductory physics textbooks.

Evaluation of the significant figure rules

As the disappearance of the significant figure rules at this point is highly unlikely, the question is: What do we do about significant figures? An evaluative look at our current usage of significant figures in both homework problems and the laboratory suggests that we can learn a lot from the past.

A. Significant figures in homework problems

Currently, one purpose for including the significant figure rules in introductory texts is so that students can determine the correct precision for a calculation in a homework problem.[1-4] In looking back at the historical context and usage of significant figures, this seems odd for several reasons. The first reason is that a number in a homework problem is not a measurement, as Ivey describes in his 1974 textbook, it is "invented."[26] The numbers may be invented so that correct answers can be distinguished from incorrect ones, or so that the numbers divide nicely to simplify calculations (setting the gravitational acceleration to 10 m/s^2 is a favorite among students). In both cases the precision is set for pedagogical, not experimental, reasons. The second reason is that the significant figure rules were not intended as a deterrent for 10-digit calculations, but rather as a means of quickly determining error propagation in the laboratory. If the only reason to discuss the significant figure rules is to keep students from writing 10-digit answers, as past history would suggest, then there are much easier ways to accomplish that goal. A statement that students should carry calculations to three digits is one method. Finally, the last reason is that the significant figure rules were originally designed as an aid to help students avoid "laborious calculations,"[15] not as a way to add an extra burden to students. These reasons lead me to conclude that the use of the significant figure rules in homework problems may be ill-advised.

B. Significant figures in the laboratory

The other purpose of the significant figure rules in introductory physics is to determine the precision of an experimentally calculated value in the laboratory. In this context the significant figure rules are incredibly valuable as many students are not equipped to perform the multivariable calculus that error analysis requires. Use of the rules in this setting requires two things.

First, as early authors have posited,[15] the significant figure rules are not exact, and students must realize they are using an approximation. A comparison of the significant figure rules to error analysis shows that in most cases the rules give the expected result, making it hard for students and faculty to remember that the rules are not perfect. For example, the significant figure rule for addition is that one should keep all of the digits up until the last significant figure of the value with the least precision. If we use the addition rule to find the sum of 1.1×10^2, 21.01, and 0.006, we obtain 1.3×10^2. If instead we apply error analysis to find the uncertainty of the calculation δf, given the associated errors δM_i, we conclude that $\delta f=\sqrt{\sum_i(\delta M_i)^2}=5$, the same as the uncertainty for the value 1.1×10^2. Thus, the δM_i for the value with the least precision dominates, setting the parameter for the addition rule. Similarly, the multiplication rule states that the number of significant figures in the product is equivalent to the number of figures in the value with the least number of significant figures. Using the multiplication rule the product of 1.23, 2, and 10.002 would be 2×10^2. The equivalent error analysis gives

$$\frac{\delta f}{f}=\sqrt{\sum_i\left(\frac{\delta M_i}{M_i}\right)^2},$$

or one part in four, requiring one significant figure in the calculation just as the rule states. However, there are limitations to the significant figure rules (see Online Appendix A[33] for a complete discussion), making it important to emphasize to students that the rules are approximations.

Second, to complete all of the calculations in an introductory physics laboratory, there must be a significant figure rule for cosine and sine functions. Amazingly a significant figure rule for these manipulations is present in Millikan's 1937 text,[22] though it has fallen out of use today. Using error analysis and this rule as a guide, we can formulate a general rule for angles (see Online Appendix B[33] for details). According to error analysis, the fractional error for the cosine function is about 1 part in 100 for input angles with an uncertainty of 0.5^o (Fig. 2). This translates to an approximate precision of three significant figures. Given this information, a significant figure rule for angles would be that: *Angles given to the ones, tenths, and hundredths place will produce calculated values with three, four, and five significant figures, respectively. Caution must be exercised for angles close to 0^o or 90^o.* With a significant figure rule for addition, multiplication, and trigonometric functions, students would be able to easily calculate the precision of their results in introductory physics labs, allowing more time for reflection on the physical nature of the experiment.

Still, one might question the use of the significant figure rules in the introductory laboratory when there are other ways of calculating uncertainty that do not involve multivariable calculus. A favorite method among students is the high-low method. In this method students calculate the result without uncertainty and then calculate it two more times with the intent of finding the lowest and highest possible results. However, this general method is not without its flaws; the method typically overestimates the error and can be difficult for students to implement if there are multiple variables. The significant figure rules, in contrast, do not rely on brute force and simply ask the students to find the measurement with the largest source of error—an important point no matter how you calculate uncertainty.

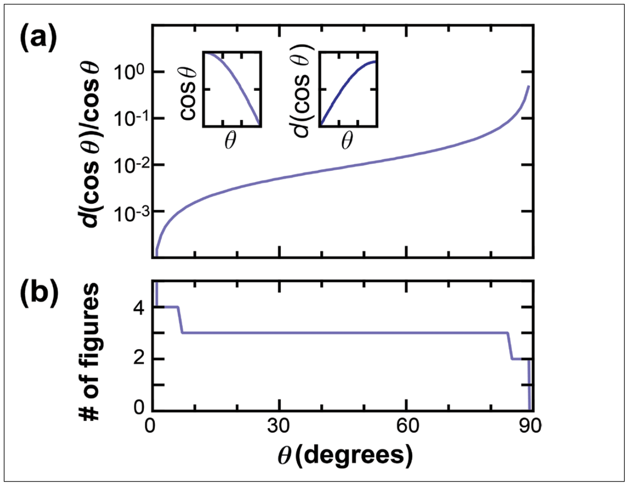

Fig. 2. Determination of the number of significant figures for the cosine function using error analysis. (a) The normalized uncertainty for the cosine function plotted against the input angle. All input angles are assumed to be precise to the one's place and have an uncertainty of 0.5^o. *Inset,* graph of the cosine function and the uncertainty in the cosine function versus angle in degrees (from 0^o to 90^o). (b) The number of significant figures for the cosine function versus input angle. For a complete discussion see the online appendix.[33]

Conclusion

In looking back at the evolution of the significant figure rules over the last 300 years, I am reminded that professors at the turn of the last century faced the same problems that we have today. Faculty would like students to realize that measurements have a precision, and that the precision of a measurement ultimately sets the precision for any calculation. If that point could be made in the laboratory, without the use of multivariable calculus that would be ideal. However, there is a disconnect as students are not given the tools to calculate the precision the whole semester. Namely there is not a significant figure rule in introductory texts for trigonometric functions. In addition, we need to emphasize that these are general rules of thumb, not rules. We should not ask our students "how many significant figures" a particular calculation has based on the rules. Instead we should ask our students "approximately what is the precision" of a particular calculation. The rules are not exact. Quizzing our students as if they are seems like a waste of time.

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